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A NOTE ON THE T-IDEAL GENERATED BY $s_3[x_1, x_2, x_3]$

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ABSTRACT
The co-characters of the T-ideal generated by the standard identity $s_3[x_1, x_2, x_3]$ are determined.

1. Introduction

Regev [4] has shown that for $n \geqq 9$, the co-character of the T-ideal K generated by the standard identity $s_3[x_1, x_2, x_3]$ over a field F of characteristic zero is $[n]+2[n-1,1]+\alpha[n-2,2]+\beta[2^2,1^{n-4}]$ where $\alpha+\beta \leqq 1$. In this note we introduce a new technique, and use it to determine the co-characters exactly.

The space V_n of multilinear polynomials of degree n in $x_1, \cdots, x_n$ may be identified with the group algebra of the symmetric group S_n over $F[3]$, and the nth co-character is then, by definition, the character of V_n, modulo the left ideal $K_n = K \cap V_n$. In fact, we shall obtain a precise description of K_n, which gives an easy test whether or not a given element of V_n is in K_n.

Since char $F = 0$, $V_n = K_n \oplus J_n$ for some left ideal J_n, and the problem is to determine the character of J_n. Since the multiplicity of a character $[\lambda]$ in V_n is $\deg[\lambda]$, the problem is equivalent to that of finding the character of K_n. We shall prove

THEOREM 1.1. (i) *For* $n \leqq 2$, $K_n = 0$.
(ii) *For* $n = 4$, $J_4 = [4]+2[3,1]+[2^2]$.
(iii) *For* $n = 3$ *or* $n \geqq 5$, $J_n = [n]+2[n-1,1]$.

The new results are those for $n \geqq 5$. The theorem is trivial for $n \leqq 3$, and we merely indicate how to apply our method to obtain another proof for the case $n = 4$.

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2. Outline of the method

DEFINITIONS. Let $y_1, \cdots, y_n$ be n *commuting* variables. Define the FS_n-modules $M^{(n-1,1)}$ and $M^{(n-2,2)}$ by

$$M^{(n-1,1)} = \mathrm{Sp}_F\{y_i \mid 1 \leqq i \leqq n\}. \text{ For } \pi \in S_n, \text{ let } \pi y_i = y_{\pi(i)}.$$

$$M^{(n-2,2)} = \mathrm{Sp}_F\{y_i y_j \mid 1 \leqq i < j \leqq n\}. \text{ Let } \pi y_i y_i = y_{\pi(i)} y_{\pi(j)}.$$

Then $M^{(n-1,1)}$ and $M^{(n-2,2)}$ are the permutation modules of S_n on the Young subgroups $S_{n-1} \times S_1$ and $S_{n-2} \times S_2$ respectively. It is well-known that their characters are $[n]+[n-1,1]$ and $[n]+[n-1,1]+[n-2,2]$ respectively.

Let $W^{(n-1,1)}$ be the submodule of $M^{(n-1,1)}$ generated by $y_2 - y_1$, and $W^{(n-2,2)}$ be the submodule of $M^{(n-2,2)}$ generated by $y_3y_4 - y_1y_4 - y_2y_3 + y_1y_2$. These are *Specht modules* [5] for the partitions $(n-1,1)$ and $(n-2,2)$, and have characters $[n-1,1]$ and $[n-2,2]$.

Of course, a Specht module W^λ can be defined for each partition λ of n; its character is $[\lambda]$.

It is elementary that the multiplicity of $[\lambda]$ in an ideal L of the group algebra of S_n is $\dim \mathrm{Hom}_{FS_n}(L, W^\lambda)$. Thus, to determine the multiplicity of $[\lambda]$ in J_n, it is sufficient to calculate $\dim \mathrm{Hom}_{FS_n}(J_n, W^\lambda)$, and this is what we shall do. Slight variations are employed to ease the numerical calculations.

3. The proofs

We start by giving an alternative proof of a known result to illustrate the new technique in action.

LEMMA 3.1. *For $n \geqq 3$, the multiplicity of $[n]$ in J_n is* 1, *and the multiplicity of $[n-1,1]$ in J_n is* 2.

PROOF. For $1 \leqq i \leqq n$, define $\theta_i \in \mathrm{Hom}_{FS_n}(V_n, M^{(n-1,1)})$ by

$$\theta_i(x_{\pi(1)} \cdots x_{\pi(n)}) = y_{\pi(i)}.$$

Thus, θ_i "isolates the ith term in the monomial $x_{\pi(1)} \cdots x_{\pi(n)}$."

Now, Lemma 4.1 of [2] implies that the general element of K_n is a linear combination of terms of the form $as_3[h_1, h_2, h_3]b$, where a, b, h_1, h_2, h_3 are monomials in some of the indeterminates $x_1, \cdots, x_n$ and $ah_1h_2h_3b \in V_n$. Hence

$$K_n \subseteq \mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_n. \tag{3.2}$$

But $x_1x_2 \cdots x_{n-1}x_n - x_1x_2 \cdots x_nx_{n-1} \in \mathrm{Ker}\,\theta_1 \backslash \mathrm{Ker}\,\theta_n$ since $n \geqq 3$. Therefore $\mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_n \neq \mathrm{Ker}\,\theta_1$. Similarly, $\mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_n \neq \mathrm{Ker}\,\theta_n$. Now, $V_n/\mathrm{Ker}\,\theta_i \cong$

$\operatorname{Im}\theta_i = M^{(n-1,1)}$ and the character of this is $[n]+[n-1,1]$. Since $[n]$ is a factor of V_n only once, it follows that

(3.3) For $n \geqq 3$, the co-character of $\operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n$ is $[n]+2[n-1,1]$.

By (3.2) and (3.3), the multiplicity of $[n]$ in J_n is 1, and that of $[n-1,1]$ is at least 2.

Since the multiplicity of $[n-1,1]$ in V_n is $n-1$, to complete the proof it is sufficient to exhibit $n-3$ linearly independent elements of $\operatorname{Hom}_{FS_n}(K_n, W^{(n-1,1)})$. Nothing is left to prove if $n=3$, so we may assume $n \geqq 4$.

$\theta_2, \cdots, \theta_{n-2}$ may be regarded as elements of $\operatorname{Hom}_{FS_n}(K_n, M^{(n-1,1)})$, and since we know that $[n]$ is not a factor of K_n, $\theta_2, \cdots, \theta_{n-2}$ in fact belong to $\operatorname{Hom}_{FS_n}(K_n, W^{(n-1,1)})$.

For $n \geqq 4$, let $v_i = x_5 x_6 \cdots x_{4+i} s_3[(x_1x_2), x_3, x_4] x_{4+i+1} \cdots x_n$.

Consider first the case when $n=4$. Then

$$v_0 = (x_1x_2)x_3x_4 - (x_1x_2)x_4x_3 + x_4(x_1x_2)x_3 - x_4x_3(x_1x_2) + x_3x_4(x_1x_2) - x_3(x_1x_2)x_4$$

and

$$\theta_2(v_0) = y_2 - y_2 + y_1 - y_3 + y_4 - y_1 = y_4 - y_3, \tag{3.4}$$

$$\theta_1(v_0) = 0. \tag{3.5}$$

For $n \geqq 4$, suppose $2 \leqq i \leqq n-2$. Then

$$(3.4) \Rightarrow \theta_i(v_{i-2}) = y_4 - y_3, \qquad \text{and}$$

$$(3.5) \Rightarrow \theta_j(v_{i-2}) = 0 \quad \text{for} \quad 1 \leqq j \leqq i-1.$$

Hence $\theta_2, \cdots, \theta_{n-2}$ are $n-3$ linearly independent elements of $\operatorname{Hom}_{FS_n}(K_n, W^{(n-1,1)})$, as required.

Replacing $M^{(n-1,1)}$ by $M^{(n-1,1)} \otimes W^{(1^n)}$ (which has character $[1^n]+[2,1^{n-2}]$), one can exhibit for $n \geqq 4$ in a very similar way n linearly independent elements of $\operatorname{Hom}_{FS_n}(K_n, M^{(n-1,1)} \otimes W^{(1^n)})$. Hence

LEMMA 3.6. *For $n \geqq 4$, the multiplicity of each of $[1^n]$ and $[2,1^{n-2}]$ in J_n is zero.*

Next, for $n \geqq 4$ and $1 \leqq i < j \leqq n$, define $\theta_{ij} \in \operatorname{Hom}_{FS_n}(V_n, M^{(n-2,2)})$ by

$$\theta_{ij}(x_{\pi(1)} \cdots x_{\pi(n)}) = y_{\pi(i)} y_{\pi(j)}.$$

By considering the action on $x_1 \cdots x_n$, it is clear that $\{\theta_{ij} \mid 1 \leqq i < j \leqq n\}$ is a linearly independent set. Since $\dim M^{(n-2,2)} = \binom{n}{2}$, we have constructed a basis of $\operatorname{Hom}_{FS_n}(V_n, M^{(n-2,2)})$. It is now possible to prove the crucial result:

LEMMA 3.7. *The multiplicity of* [3,2] *in* J_5 *is zero.*

PROOF. Let $u_1 = s_3[(x_1x_2), x_3, x_4]x_5$, $u_2 = s_3[(x_1x_2), (x_3x_4), x_5]$ and $u_3 = s_3[(x_1x_2x_3), x_4, x_5]$. Tables I, II and II record the images $\theta_{ij}(u_k)$ for $k = 1,2$ and 3 respectively. (For example, the second row of Table I states that $\theta_{13}(u_1) = 2y_1y_3 - 2y_1y_4 - y_2y_3 + y_2y_4$.)

TABLE I

Suffices for y_iy_j →	12	13	14	15	23	24	25	34	35	45
θ_{12}		−1	1							
θ_{13}		2	−2		−1	1				
θ_{14}		−1	1		1	−1				
θ_{15}										
θ_{23}		−1	1		1	−1				
θ_{24}		1	−1		−2	2				
θ_{25}									−1	1
θ_{34}					1	−1				
θ_{35}									1	−1
θ_{45}										

TABLE II

	12	13	14	15	23	24	25	34	35	45
θ_{12}				1					−1	
θ_{13}				−1			1		1	−1
θ_{14}			1	−1	−1				1	
θ_{15}			−1	1	1		−1		−1	1
θ_{23}	1		−1		1		−1	−1		1
θ_{24}			1		−1					
θ_{25}			1		−1		1			−1
θ_{34}	−1		−1	1	1			1	−1	
θ_{35}				−1			1		1	−1
θ_{45}							−1			1

TABLE III

	12	13	14	15	23	24	25	34	35	45
θ_{12}			-1	1						
θ_{13}			1	-1		-1	1			
θ_{14}			1	-1		1	-1	-1	1	
θ_{15}			-1	1				1	-1	
θ_{23}			-1	1						
θ_{24}										
θ_{25}			1	-1		-1	1	-1	1	
θ_{34}								1	-1	
θ_{35}						1	-1	-1	1	
θ_{45}								1	-1	

TABLE IV

1	1	
1	1	
1	1	
1	1	1
1		
1		
1		1
1		
1		1
1		1

Omitted numbers are zero. It is straightforward to check that the columns of the tables span a space of dimension 7, and that Table IV gives 3 columns which are orthogonal to all these. If $\theta \in \mathrm{Hom}_{FS_n}(V_5, M^{(3,2)})$ and $\mathrm{Ker}\,\theta$ contains K_5, it follows that θ must have the form

$$\theta = \alpha(\theta_{12} + \theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} + \theta_{34} + \theta_{35} + \theta_{45})$$

$$+ \beta(\theta_{12} + \theta_{13} + \theta_{14} + \theta_{15}) + \gamma(\theta_{15} + \theta_{25} + \theta_{35} + \theta_{45}), \qquad \text{with} \quad \alpha, \beta, \gamma \in F.$$

Thus

$$\theta(x_1x_2x_3x_4x_5) = \alpha(y_1y_2 + y_1y_3 + \cdots + y_4y_5)$$

$$+ \beta(y_1y_2 + y_1y_3 + y_1y_4 + y_1y_5) + \gamma(y_5y_1 + y_5y_2 + y_5y_3 + y_5y_4).$$

Therefore, $\mathrm{Im}\,\theta$ is a homomorphic image of $M^{(5)} \oplus M^{(4,1)} \oplus M^{(4,1)}$ and does not contain $W^{(3,2)}$.

Now, if J_5 contained a left ideal isomorphic to $W^{(3,2)}$, we could take the projection onto this ideal, and produce an element of $\mathrm{Hom}_{FS_n}(V_5, M^{(3,2)})$ having K_5 in its kernel and $W^{(3,2)}$ in its image. As we have shown that this is impossible, the lemma is proved.

NOTE. An alternative proof of the last lemma is to take the standard basis $e_1, \cdots, e_5$ for $W^{(3,2)}$ and define $\varphi_i \in \mathrm{Hom}(V_5, W^{(3,2)})$ by $\varphi_i(x_1 \cdots x_5) = e_i$. Then check that the only linear combination of $\varphi_1, \cdots, \varphi_5$ sending u_1, u_2 and u_3 to zero is 0. Although this proof involves spaces of dimension 5 and not 10, the numerical calculations take longer.

Considering $\mathrm{Hom}_{FS_n}(V_5, M^{(3,2)} \otimes W^{(1^5)})$, a similar proof gives

LEMMA 3.8. *The multiplicity of* $[2^2, 1]$ *in* J_5 *is zero.*

(Regev [4] gives an alternative method of proving this.)

Somewhat easier to check is that

$$\{\theta \mid \theta \in \mathrm{Hom}_{FS_4}(V_4, M^{(2,2)}) \text{ and } K_4 \subseteq \mathrm{Ker}\,\theta \text{ and } \mathrm{Im}\,\theta \supseteq W^{(2,2)}\}$$

is spanned by $\theta_{12} - \theta_{23} + \theta_{34}$. Therefore J_4 has precisely one factor isomorphic to $W^{(2,2)}$. This, together with Lemmas 3.1 and 3.6 proves the second part of Theorem 1.1.

We now finish the proof of Theorem 1.1, using a difficult result from Regev [4]; in the next section, we show how to complete the proof without quoting Regev's paper.

The proof of Regev's theorem 3.13 gives the following inequality between the dimensions c_n of J_n:

(3.9) $$c_n \leqq n + c_{n-1} \quad \text{for} \quad n \geqq 3.$$

Since $c_4 = 9$ (Theorem 1.1 (ii)), we have $c_5 \leqq 14$. But our results so far show that $J_5 = [5] + 2[4,1] + \alpha[3,1^2]$ (with α an integer). Therefore $c_5 = 1 + 8 + \alpha 6$, whence $\alpha = 0$, proving Theorem 1.1 for the case $n = 5$.

If $n \geqq 6$, $J_n = [n] + 2[n-1,1] + \chi_n$, say. Induction and the inequality (3.9) give $\deg \chi_n \leqq n - 2$. But χ_n contains neither $[n]$ nor $[1^n]$, and for $n \geqq 5$ the smallest irreducible degree of S_n (other than 1) is $n - 1$ (Burnside [1], appendix). Therefore $\chi_n = 0$, as required.

The following corollary of Theorem 1.1 gives a simple test of whether or not a given element of V_n belongs to K_n:

COROLLARY 3.10. (i) *For* $n = 3$ *or* $n \geqq 5$, $K_n = \mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_n$.
(ii) $K_4 = \mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_4 \cap \mathrm{Ker}(\theta_{12} - \theta_{23} + \theta_{34})$.

PROOF. For $n \neq 4$, the corollary comes from (3.2) and (3.3). We have explained above why the extra Kernel arises when $n = 4$.

In particular, we have the surprising

COROLLARY 3.11. $x_1x_2x_3x_4x_5 - x_1x_2x_4x_3x_5$ *and* $x_1x_2x_3x_4x_5 - x_1x_3x_2x_4x_5 \in K$.

PROOF. Both elements are in $\mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_5$, and so belong to K_5.

4. A basis for K_n $(n \geqq 5)$

In this section, we construct a basis for $\mathrm{Ker}\,\theta_1 \cap \mathrm{Ker}\,\theta_n$. In view of Corollary 3.10, this will be a basis for K_n when $n \geqq 5$. The basis provides an alternative proof for Theorem 1.1.

DEFINITION. Put a partial order $\trianglerighteq$ on the set of monomials in V_n by

$$x_{\sigma(1)}\cdots x_{\sigma(n)} \trianglerighteq x_{\tau(1)}\cdots x_{\tau(n)} \text{iff for all } j \sum_{i=1}^{j} \sigma(i) \leqq \sum_{i=1}^{j} \tau(i).$$

This is undoubtedly the "correct" ordering on monomials (we use the symbol $\trianglerighteq$ by analogy with the standard notation for the dominance order on Young disgrams), but the reader may prefer to select a total lexicographic order which contains " $\trianglerighteq$ ".

Assume $n \geqq 4$.

Let

$$P_1^n = \{\pi \in S_n \mid \exists 1 < i < j < n \quad \text{with} \quad \pi(i) > \pi(j)\} \quad \text{and}$$

$$P_2^n = \{\pi \in S_n \mid \pi(1) > \pi(2) < \pi(3) < \cdots < \pi(n-1) > \pi(n)\}.$$

An easy calculation gives $|P_1^n| = n! - n(n-1)$ and $|P_2^n| = n - 3 + (n-2)^2$.

For $\pi \in P_1^n$, define e_π as follows. Choose $1 < i < j < n$ with $\pi(i) > \pi(j)$ and let

$$e_\pi = x_{\pi(1)}\cdots x_{\pi(i)}\cdots x_{\pi(j)}\cdots x_{\pi(n)} - x_{\pi(1)}\cdots x_{\pi(j)}\cdots x_{\pi(i)}\cdots x_{\pi(n)}$$

[$= x_{\pi(1)}\cdots x_{\pi(n)}(1_{S_n} - (i,j))$, recalling that right multiplication by an element of S_n effects a *place* permutation].

For $\pi \in P_2^n$, define e_π by

$$e_\pi = x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n-1)}x_{\pi(n)} - x_{\pi(2)}x_{\pi(1)}\cdots x_{\pi(n-1)}x_{\pi(n)}$$

$$- x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}x_{\pi(n-1)} + x_{\pi(2)}x_{\pi(1)}\cdots x_{\pi(n)}x_{\pi(n-1)}$$

[$= x_{\pi(1)}\cdots x_{\pi(n)}(1_{S_n} - (1,2))(1_{S_n} - (n-1, n))$].

Now, $\{e_\pi \mid \pi \in P_1^n \cup P_2^n\} \subseteq \operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n$. But e_π involves x_π $(:= x_{\pi(1)}\cdots x_{\pi(n)})$ and the other monomials x_τ involved in e_π satisfy $\tau \vartriangleright \pi$. Since the π's are all different, we have constructed a linearly independent subset of $\operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n$. But $\dim(\operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n) = n! - 1 - 2(n-1)$, by (3.3), $= |P_1^n| + |P_2^n|$, and so we have obtained a *basis*. (An easy alternative way of seeing that we have a spanning set is to check that a last monomial (in the $\vartriangleright$ order) involved in an element of $\operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n$ must belong to $P_1^n \cup P_2^n$.)

Another proof of Theorem 1.1 goes as follows. First verify that $[3, 1^2]$ is not in J_5, either by using the techniques above, or utilizing the note added in proof by Regev [4]. Then J_5 must be $[5] + 2[4, 1]$, and $K_5 = \operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_5$.

Assume $n \geqq 5$ and $K_n = \operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_n$. Then for $\pi \in S_n$,

$$x_{\pi(1)}\cdots x_{\pi(i)}\cdots x_{\pi(j)}\cdots x_{\pi(n)} - x_{\pi(1)}\cdots x_{\pi(j)}\cdots x_{\pi(i)}\cdots x_{\pi(n)} \in K.$$

Therefore,

$$x_{\pi(1)} \cdots x_{\pi(i)} \cdots x_{\pi(j)} \cdots x_{\pi(n)} x_{n+1} - x_{\pi(1)} \cdots x_{\pi(j)} \cdots x_{\pi(i)} \cdots x_{\pi(n)} x_{n+1}$$

and

$$x_{n+1} x_{\pi(1)} \cdots x_{\pi(i)} \cdots x_{\pi(j)} \cdots x_{\pi(n)} - x_{n+1} x_{\pi(1)} \cdots x_{\pi(j)} \cdots x_{\pi(i)} \cdots x_{\pi(n)}$$

belong to K. Hence, renumbering the variables, K_{n+1} contains all elements of the form e_π with π in P_1^{n+1}.

Also, for $\pi \in S_{n+1}$,

$$x_{\pi(1)}x_{\pi(2)} \cdots (x_{\pi(n-1)}x_{\pi(n)})x_{\pi(n+1)} - x_{\pi(2)}x_{\pi(1)} \cdots (x_{\pi(n-1)}x_{\pi(n)})x_{\pi(n+1)}$$

$$- x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n+1)}(x_{\pi(n-1)}x_{\pi(n)}) + x_{\pi(2)}x_{\pi(1)} \cdots x_{\pi(n+1)}(x_{\pi(n-1)}x_{\pi(n)}),$$

$$x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n+1)}x_{\pi(n-1)}x_{\pi(n)} - x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n-1)}x_{\pi(n+1)}x_{\pi(n)}$$

and

$$x_{\pi(2)}x_{\pi(1)} \cdots x_{\pi(n-1)}x_{\pi(n+1)}x_{\pi(n)} - x_{\pi(2)}x_{\pi(1)} \cdots x_{\pi(n+1)}x_{\pi(n-1)}x_{\pi(n)}$$

all belong to K. Adding these, we get

$$x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n-1)}x_{\pi(n)}x_{\pi(n+1)} - x_{\pi(2)}x_{\pi(1)} \cdots x_{\pi(n-1)}x_{\pi(n)}x_{\pi(n+1)}$$

$$- x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n-1)}x_{\pi(n+1)}x_{\pi(n)} + x_{\pi(2)}x_{\pi(1)} \cdots x_{\pi(n-1)}x_{\pi(n+1)}x_{\pi(n)}$$

belongs to K. In particular, K contains all elements of the form e_π with π in P_2^{n+1}. Therefore, K_{n+1} contains a basis of $\operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_{n+1}$ and by (3.2) $K_{n+1} = \operatorname{Ker}\theta_1 \cap \operatorname{Ker}\theta_{n+1}$. By induction, and (3.3), Theorem 1.1 is proved.

Finally, it should be noted that Conjecture 2 of Regev [4] is false for $d = 3$ and $n \geqq 4$, since $x_1x_2 \cdots x_{n-1}x_nx_{n+1} - x_1x_2 \cdots x_nx_{n-1}x_{n+1} \in K_{n+1}$ (by Corollary 3.10) but $x_1x_2 \cdots x_{n-1}x_n - x_1x_2 \cdots x_nx_{n-1} \notin \operatorname{Ker}\theta_n \supseteq K_n$.

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